From Barycentric Coordinates to Whitney Forms:

Turn Your Mesh into a Computational Structure

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Big Picture

Defining Barycentric Coordinates

- super simple, you already know them

Entending them to Edges, Faces, Tets

- called Whitney basis functions

Deriving a whole *Discrete* Calculus

- as the one you know, but on mesh

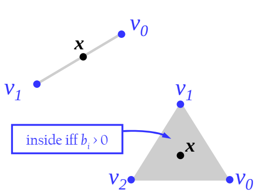
Scared?

- **Don't be**: *it's only numbers on mesh elmts!*

Barycentric Coordinates

You already know barycentric coordinates:

- given (d+1) point in R^d,
- way to *locate* point within their convex hull



$$x = \sum_{i=0}^{d} b_i[x]\, v_i$$

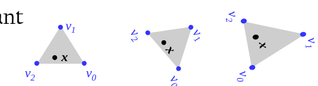
with $\sum_{i=0}^{d} b_i[x] = 1$

$\in R^d$

What's the Point?

Nice, important properties

- Möbius invented it in 1827 – "mass points"
  - under Gauß's supervision–got to be good
- coordinate-free geometry
  - global vs. relative positioning
  - storing numbers on vertices
  - intrinsic; indpt on dim. of ambient space
  - affine invariant



Computing Barycentric Coords

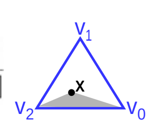
Question:

*Given a **d-simplex**, find the bary. coords $b_i[x]$ **of an inside point x** for each vertex v_i such as:*

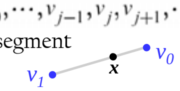
$$b_i[x] \geq 0 \qquad \sum_i b_i[x] = 1 \qquad \sum_i v_i\, b_i[x] = x$$

Unique solution (in any dim.):

$$b_j[x] = \frac{Vol[v_0, \cdots, v_{j-1}, x, v_{j+1}, \cdots, v_d]}{Vol[v_0, \cdots, v_{j-1}, v_j, v_{j+1}, \cdots, v_d]}$$

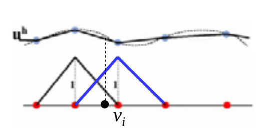


Try it on a segment



Barycentric Basis Functions

In 1D:

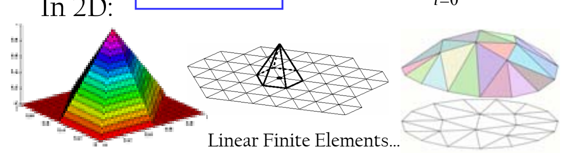


$$x = \sum_{i=0}^{d} b_i[x]\, v_i$$

linear interpolation

$$u(x) = \sum_{i=0}^{d} b_i[x]\, u_i$$

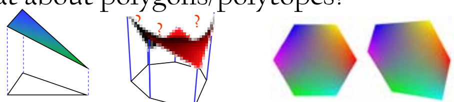
In 2D:



Linear Finite Elements...

Extending it to Other Shapes?

What about polygons/polytopes?

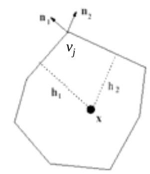


Barycentric coordinates for **polytopes**

- ouch, not unique anymore…
- additional requirements?
  - smoothness of basis functions
  - tensor product (square→bilinear)
  - face restriction (should fit (n-1)D bc)
  - simplicity of evaluation…

Generalyzed B. Coords [Warren *et al* 04]

For convex polytope, simple expression



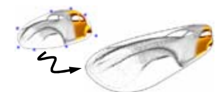
$b_i[x] = w_i[x] / \sum w_j[x]$, with:

$$w_j(\mathbf{x}) = \frac{|\det(\mathbf{n}_{j,1}, \dots, \mathbf{n}_{j,n})|}{\prod_{k=1}^{n} h_{j,k}(\mathbf{x})}$$

volume of normal cone

prod. of distances to adj. faces

Rational basis fcts of degree (F–n)
(because zero on (F–n) lines)
that extends to smooth domains!



Other Types of GBC

Other approaches:

- mean value coordinates in 2D and 3D
  - Floater, 2003/2005, positive for *any* star-shaped polygons
- Sibson's – a bit complicated to compute
- blend of various coordinates
  - Hormann *et al.*, 2003
- for any 2D polygon [Malsch *et al.*]
- See new paper by Ju *et al.* this year

None reproduces tensor prod. or generalizes to nD

So What?

What did we accomplish?

- discrete scalar fields
  - discrete sampling = values on nodes of a mesh
  - spatial interpolation to allow arbitrary evals

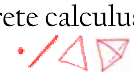
Where to Go from Here?

- what about vector fields?
- what about computations?
  - gradient, curl, div, laplacian, you-name-it
- keeping them *mesh-intrinsic* all the way?

Intrinsic Calculus on Meshes

We can bootstrap a whole discrete calculus!

- using only values on simplices



- and if needed, interpolation in space
- preview:

point-based scalar field —∇ / d→ *edge-based* vector field —$\nabla\times$ / d→ *face-based* vector field —$\nabla\cdot$ / d→ *cell-based* scalar field

- deep roots in mathematics
  - algebraic topology, differential geometry
- *but very simple to implement and use*

Discrete Differential Quantities

Hinted in the talks before…

- they "live" at special places, *as distributions*
  - Gaussian curvature at vertices ONLY
  - mean curvature at edges ONLY
- they can be handled through integration
  - integration calls for k-*forms* (antisymmetric tensors)
    - objects that beg to be integrated (ex: $\int f(x)\,dx$)
  - k-forms are evaluated on kD set
    - 0-form is evaluated at a point, 1-form at a curve, etc…

that's a 1-form

Forms You Know For Sure

Scalar functions: 0-forms

Digital Images: 2-forms

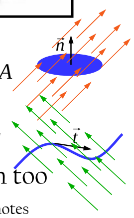
- incident flux on sensors (W/m^2)

Magnetic Field ***B***: 2-form

- only measurement possible: $\iint \vec{B}.\vec{n}\, dA$
- any physical flux is a 2-form too

Elctrical Force *E*: 1-form $\int \vec{E}.\vec{t}\, dl$

- any physical circulation is a 1-form too



notion of pseudo forms—see notes

Exterior Calculus of Forms

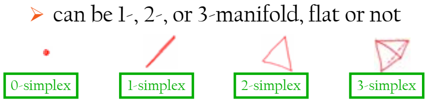
Foundation of calculus on smooth manifolds

- Historically, purpose was to extend div/curl/grad
  - Poincaré, Cartan, Lie, ...
- Basis of differential and integral computations
  - highlights topological and geometrical structures
  - modern diff. geometry, Hodge decomposition, ...
- A hierarchy of basic operators are defined:
  - d, *, $\wedge$, $\flat$, $\sharp$, i_X, $\mathcal{L}_X$
- See [Abraham, Marsden, Ratiu] , ch. 6-7

Discrete Exterior Calculus

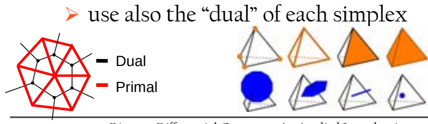
Simplicial complex only spatial structure

- can be 1-, 2-, or 3-manifold, flat or not



Idea: *Sampling Forms on Each Simplex!*

- "extends" the idea of point-sampling
- use also the "dual" of each simplex



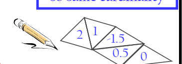
Discrete Differential Forms

Discrete k-form = values on each kD set

- primal discrete k-form: value on *each* k-simplex
- dual discrete k-form: value on *each* dual (n-k)-cell
- the rest is defined through linearity

dual of k-simplex, so same cardinality

$$\int_{\bigcup_j \sigma_j} \omega = \sum_j \int_{\sigma_j} \omega$$

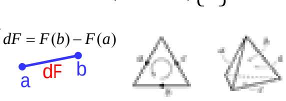


- in math terms: chains pair with cochains
  (natural pairing = integration)
- in CS terms: k-form = vector of values

Notion of Exterior Derivative

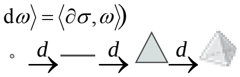
- Stokes/Green/{...} theorem: $\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega$

$\int_a^b dF = F(b) - F(a)$



turns a integral into a boundary integral

- so d and ∂ are dual $\langle\langle \sigma, \mathrm{d}\omega \rangle = \langle \partial\sigma, \omega \rangle\rangle$
- Implementation?

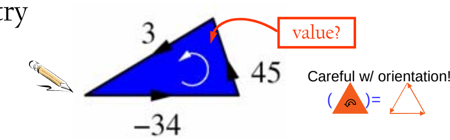


  - As simple as an incidence matrix!
  - ex: d(1-form)=incidence matrix of edges & faces

    Bean counting: [|F|x|E|] (|E|) = (|F|)

Exterior Derivative

Let's try



- No "metric" needed! (no size measurement)
- Try d, then d on an arbitrary form...
  - zero; why?
  - because $\partial \circ \partial = 0$
  - good: div(curl)=curl(grad)=0 automatically

Hodge Star

Take forms to dual complex *(and vice-versa)*

- switch values btw primal/dual

$$\star : \Omega^p \to \Omega_\star^{n-p}$$

- "diagonal" hodge star

$$\frac{1}{|\star\sigma^p|}\int_{\star\sigma^p}\star\alpha^p = \frac{1}{|\sigma^p|}\int_{\sigma^p}\alpha^p$$

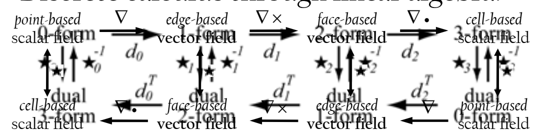
  - again, a simple (diagonal) matrix

common "average" value

- now the metric enters…
- Hodge star defines accuracy
  - order of approximation of the metric

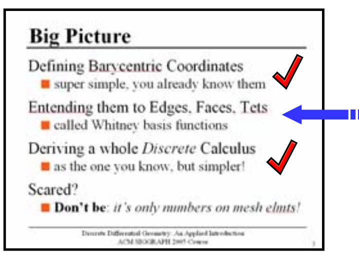
Discrete deRham Complex

Discrete calculus through linear algebra:



- simple exercise in matrix assembly
- all made out of two trivial operations:
  - summing values on simplices (d/∂)
  - scaling values based on local measurements (⋆)

A Step Back



Big Picture

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- super simple, you already know them

Entending them to Edges, Faces, Tets
- called Whitney basis functions

Deriving a whole *Discrete* Calculus
- as the one you know, but simpler!

Scared?
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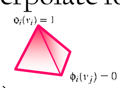
Interpolating Discrete Forms

Whitney basis fcts to interpolate forms

- 0-forms (functions)
  - "hat" functions



- 1-forms (edge elements!)
  - Whitney forms: $\phi_{ij} = \phi_i d\phi_j - \phi_j d\phi$
  - basis of 1-forms since $\phi_{ij}(e_{kl}) = 0$ on edges ≠ (i,j)

think ∇ for computations

- 2-forms: face elmts

also a fct of ϕ and $\nabla\phi$

- 3-forms: constant per tet

Higher order bases for smoother interp.

What We'll be Able To Do

Why DEC is New

Not quite like:

- Finite Elements
  - use nodes and cells only
  - tries to enforce local relationships *globally well*
- Finite Differences
  - sorta local polynomial fitting, loses invariants
- Finite Volumes
  - use cells and nodes only
  - good at local relations, often bad at global ones

Why DEC is Limited

It Does Not Substitute For:

- Numerical Analysis
  - accuracy and convergence rate still need careful study
- Good Meshing Tools
  - bad mesh? bad results, guaranteed...
  - stay tuned; we'll address the issue in the last talk
- Good hacking skills
  - see *Building your own DEC at home* in notes

Why DEC is Good

If You Ask Me:

- basic discrete operators, consistently derived
  - easy to compute given a discrete mesh
  - separating topology from geometry
    - helps narrowing down where accuracy is lost
  - conservation laws can be preserved *exactly*
- preserving structures at the discrete level!
  - applicable to a variety of problems
  - good foundations for further studies
- can be used as basis for 'simplex sampling'

Other Related Research Areas

Variational Integrators

- principle of *least-action* is crucial
  - motion is a geodesic if we use action as "metric"
  - preserve *invariants and symmetries*
- "*not accurate*?": urban legend, simply untrue

Discrete Differential Euclidean Geometry

- it all ties up through the use of connections
- but it will be for another day...

Geometric Algebra

Take-Home Message

Geometric Approach to Computations

- discrete setup acknowledged from the get-go
- choice of proper habitat for quantities
- whole calculus built using only:
  - boundary of mesh elements
  - scaling by local measurements
- preserving *structural indentities*
  - they are not just abstract concepts: they represent *defining symmetries*